

THE SIDON CONSTANT FOR HOMOGENEOUS POLYNOMIALS

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ABSTRACT. The Sidon constant for the index set of nonzero m -homogeneous polynomials P in n complex variables is the supremum of the ratio between the ℓ^1 norm of the coefficients of P and the $H^\infty(\mathbb{D}^n)$ norm of P . We present an estimate which gives the right order of magnitude for this constant, modulo a factor depending exponentially on m . We use this result to show that the Bohr radius for the polydisc \mathbb{D}^n is bounded from below by a constant times $\sqrt{(\log n)/n}$.

1. INTRODUCTION

This note presents an estimate on the Sidon constant $S(m, n)$ for the index set of homogeneous polynomials of degree m in n complex variables. The result is optimal in the sense that the exact value of $S(m, n)$ is determined up to a factor depending exponentially on m . We will use this estimate to find the precise asymptotic behavior of the n -dimensional Bohr radius, which was introduced and studied by H. Boas and D. Khavinson [2].

The Sidon constant $S(m, n)$ for the index set $\{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) : |\alpha| = m\}$ is defined in the following way. Let

$$P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$$

be a homogeneous polynomial of degree m in n complex variables. We let \mathbb{D}^n denote the unit polydisc in \mathbb{C}^n and set

$$\|P\|_\infty = \sup_{z \in \mathbb{D}^n} |P(z)| \quad \text{and} \quad \|P\|_1 = \sum_{|\alpha|=m} |c_\alpha|;$$

then $S(m, n)$ is the smallest constant C such that the inequality $\|P\|_1 \leq C\|P\|_\infty$ holds for every P . It is plain that $S(1, n) = 1$ for all n , and this case is therefore excluded from our discussion. Our main result is the following estimate.

Theorem 1. *There exists an absolute constant C such that the Sidon constant $S(m, n)$ satisfies*

$$(1) \quad S(m, n) \leq C^m \sqrt{\frac{n^{m-1}}{(m-1)!}}$$

when $n > m^2 > 1$.

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The Sidon constant $S(m, n)$ is effectively the same as the unconditional basis constant for the monomials of degree m in $H^\infty(\mathbb{D}^n)$; the latter is larger than $S(m, n)$ by a factor not exceeding 2. This and similar unconditional basis constants were studied in [6], where it was established that $S(m, n)$ is bounded from above and below by $n^{(m-1)/2}$ times constants depending only on m . The more precise estimate

$$(2) \quad S(m, n) \leq C^m n^{\frac{m-1}{2}},$$

with C an absolute constant, can be extracted from [7]. By Hölder's inequality, (2) also follows from an interesting inequality of H. Queffélec [11], which says that the $\ell^{2m/(m+1)}$ norm of the coefficients of a homogeneous polynomial P of degree m is bounded by $\|P\|_\infty$ times a certain precise constant depending only on m . A more direct deduction of (2) is implicit in the work of F. Bohnenblust and E. Hille [3]; this approach has inspired our proof of (1).

Note that we also have the following trivial estimate:

$$(3) \quad S(m, n) \leq \sqrt{\binom{n+m-1}{m}},$$

which is a consequence of the Cauchy–Schwarz inequality along with the fact that the number of different monomials of degree m in n variables is $\binom{n+m-1}{m}$. Comparing (1) and (3), we see that our estimate gives a nontrivial result only in the range $\log n > m$. Using the Salem–Zygmund inequality for random trigonometric polynomials (see [10, p. 68]), one may check that the estimates (3) and (1) together give the right value for $S(m, n)$, up to a factor less than c^m with $c < 1$ an absolute constant.

Our application of Theorem 1 to the asymptotic behavior of the Bohr radius for the polydisc will further illuminate the significance of (1). Following [2], we now let K_n be the n -dimensional Bohr radius, i.e., the largest positive number r such that all polynomials $\sum_\alpha c_\alpha z^\alpha$ satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_\alpha |c_\alpha z^\alpha| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_\alpha c_\alpha z^\alpha \right|.$$

The classical Bohr radius K_1 was studied and estimated by H. Bohr [4] himself, and it was shown independently by M. Riesz, I. Schur, and F. Wiener that $K_1 = 1/3$. In [2], the two inequalities

$$(4) \quad \frac{1}{3} \sqrt{\frac{1}{n}} \leq K_n \leq 2 \sqrt{\frac{\log n}{n}}$$

were established for $n > 1$. The paper of Boas and Khavinson aroused new interest in the Bohr radius and has been a source of inspiration for many subsequent papers. For some time (see for instance [1]) it was thought that the left-hand side of (4) could not be improved. However, using (2), A. Defant and L. Frerick [7] showed that

$$K_n \geq c \sqrt{\frac{\log n}{n \log \log n}}$$

holds for some constant $c > 0$.

Using Theorem 1, we will prove the following estimate.

Theorem 2. *The n -dimensional Bohr radius K_n satisfies*

$$K_n \geq \gamma \sqrt{\frac{\log n}{n}}$$

for an absolute constant $\gamma > 0$.

Combining this result with the right inequality in (4), we conclude that

$$K_n = b(n) \sqrt{\frac{\log n}{n}}$$

with $0 < \gamma \leq b(n) \leq 2$. It is possible to extract from our methods a numerical value for γ larger than 0.2, cf. the concluding remark of Section 5.

2. PRELIMINARIES ON MULTILINEAR FORMS

The transformation of a homogeneous polynomial to a corresponding multilinear form will play a crucial role in the proof of Theorem 1. We denote by B an m -multilinear form in \mathbb{C}^n , i.e., given m points $z^{(1)}, \dots, z^{(m)}$ in \mathbb{C}^n , we set

$$B(z^{(1)}, \dots, z^{(m)}) = \sum_{\beta} b_{\beta} z_{\beta_1}^{(1)} \cdots z_{\beta_m}^{(m)}.$$

We may express the coefficients as $b_{\beta} = B(e^{\beta_1}, \dots, e^{\beta_m})$, where $\{e^i\}_{i=1, \dots, n}$ is the canonical base of \mathbb{C}^n . The form B is symmetric if for every permutation σ of the set $\{1, 2, \dots, m\}$, $B(z^{(1)}, \dots, z^{(m)}) = B(z^{(\sigma(1))}, \dots, z^{(\sigma(m))})$. If we restrict a symmetric multilinear form to the diagonal $P(z) = B(z, \dots, z)$, then we obtain a homogeneous polynomial. The converse is also true: Given a homogeneous polynomial $P : \mathbb{C}^n \rightarrow \mathbb{C}$ of degree m , by polarization, we may define the symmetric m -multilinear form $B : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \rightarrow \mathbb{C}$ by setting

$$(5) \quad B(z^{(1)}, \dots, z^{(m)}) = \frac{1}{2^m m!} \sum_{\substack{\varepsilon_i = \pm 1 \\ 1 \leq i \leq m}} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m P\left(\sum_{i=1}^m \varepsilon_i z^{(i)}\right)$$

so that $B(z, \dots, z) = P(z)$. In what follows, B will denote the symmetric m -multilinear form obtained in this way from P .

We will consider the analogous norms for symmetric multilinear forms as those introduced above. This means that we set

$$\|B\|_{\infty} = \sup_{\mathbb{D}^n \times \cdots \times \mathbb{D}^n} |B(z^{(1)}, \dots, z^{(m)})| \quad \text{and} \quad \|B\|_1 = \sum_{|\beta|=m} |b_{\beta}|.$$

It will be important for us to be able to relate the norms of P and B . It is plain that $\|P\|_{\infty} \leq \|B\|_{\infty}$. On the other hand, it was proved by L. Harris [9] that we have, for non-negative integers m_1, \dots, m_k with $m_1 + \cdots + m_k = m$,

$$(6) \quad |B(\underbrace{z^{(1)}, \dots, z^{(1)}}_{m_1}, \dots, \underbrace{z^{(k)}, \dots, z^{(k)}}_{m_k})| \leq \frac{m_1! \cdots m_k!}{m_1^{m_1} \cdots m_k^{m_k}} \frac{m^m}{m!} \|P\|_{\infty};$$

this result can be obtained from the polarization formula (5).

To compare the $\|\cdot\|_1$ norms, observe that the coefficients b_β of B can be computed from the corresponding coefficient of P : $b_\beta = c_\alpha/h(\beta)$, where $h(\beta)$ is the number of different words that can be formed with the letters in β . The corresponding α_j is the number of times any of the indices β_i equals j . It is therefore clear that

$$\sum_{\alpha} |c_{\alpha}| = \sum_{\beta} |b_{\beta}|,$$

or, in other words, $\|P\|_1 = \|B\|_1$.

3. THE TETRAHEDRAL PART OF A HOMOGENEOUS POLYNOMIAL

A polynomial $Q(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}$ is said to be tetrahedral if c_{α} is nonzero only if $\max_j \alpha_j \leq 1$; thus no term in Q contains a factor of degree 2 or higher in any of the variables z_1, \dots, z_n . Now set $E = \{(\alpha_1, \dots, \alpha_n) : |\alpha| = m, \alpha_i \leq 1, \forall i = 1, \dots, n\}$. Then $T(P) = \sum_{\alpha \in E} c_{\alpha} z^{\alpha}$ is the tetrahedral part of P and $R(P) = P - T(P)$ is the remainder corresponding to monomials containing a higher order power in at least one of the variables z_1, \dots, z_n .

In the next lemma, p_1, p_2, \dots are the prime numbers, listed by increasing order, and $\text{sinc } x = (\sin x)/x$.

Lemma 1. *We have $\|T(P)\|_{\infty} \leq \kappa^m \|P\|_{\infty}$ for every homogeneous polynomial P of degree m , where the constant κ can be taken as*

$$\kappa = \left(\prod_{k=1}^{\infty} \text{sinc} \frac{\pi}{p_k} \right)^{-1} = 2.209 \dots$$

Proof. We will need the counting function for the prime numbers, which will be denoted by $\varpi(x)$, in order not to confuse it with the number π . We begin by constructing some auxiliary functions. Set $Q = [0, 1]^{\varpi(m)}$, let $t = (t_1, \dots, t_{\varpi(m)})$ denote a point in Q , and let $d\mu$ be Lebesgue measure on Q . Define

$$r_m(t) = c_m \exp \left(2\pi i \left(\frac{t_1}{2} + \frac{t_2}{3} + \dots + \frac{t_{\varpi(m)}}{p_{\varpi(m)}} \right) \right),$$

where

$$c_m = \prod_{k=1}^{\varpi(m)} \left(\frac{p_k}{2\pi i} \left(e^{\frac{2\pi i}{p_k}} - 1 \right) \right)^{-1}.$$

The functions $r_m : Q \rightarrow \mathbb{C}$ have the following properties:

- (i) $\int_Q r_m(t) d\mu(t) = 1$,
- (ii) $\int_Q r_m^k(t) d\mu(t) = 0$ for all $k = 2, \dots, m$,
- (iii) $|r_m(t)| \leq \kappa$ for all t in Q and all $m > 1$.

It is immediate that (i) and (ii) are satisfied. We note that (iii) also holds, because $|r_m(t)| \equiv |c_m|$ and

$$|c_m|^{-2} = \prod_{k=1}^{\varpi(m)} \frac{p_k^2}{(2\pi)^2} \left| e^{\frac{2\pi i}{p_k}} - 1 \right|^2 = \prod_{k=1}^{\varpi(m)} \text{sinc}^2 \frac{\pi}{p_k}.$$

By properties (i) and (ii),

$$T(P)(z) = \int_{Q^n} P(z_1 r(t^1), \dots, z_n r(t^n)) d\mu(t^1) \cdots d\mu(t^n),$$

and so, by property (iii), $|P(z_1 r(t^1), \dots, z_n r(t^n))| \leq \kappa^m \|P\|_\infty$ for every z in \mathbb{D}^n . \square

We can similarly define a decomposition of symmetric m -multilinear forms. Let F be the set of multiindices $F = \{(\beta_1, \dots, \beta_m) : 1 \leq \beta_i \leq n \text{ and all indices } \beta_k \text{ are pairwise disjoint}\}$. Then we may decompose $B = T(B) + R(B)$, where

$$T(B)(z^{(1)}, \dots, z^{(m)}) = \sum_{\beta \in F} b_\beta z_{\beta_1}^{(1)} \cdots z_{\beta_m}^{(m)},$$

Clearly, if P is a homogeneous polynomial and B its corresponding symmetric multilinear form, then $T(P)$ has $T(B)$ as the corresponding multilinear form.

4. PROOF OF THEOREM 1

Since

$$\| \|P\|_1 = \| \|R(P)\|_1 + \| \|T(P)\|_1,$$

it will suffice to obtain appropriate estimates for each of the norms $\| \|R(P)\|_1$ and $\| \|T(P)\|_1$. The two lemmas below together give the required bound for $\| \|P\|_1$.

We begin by estimating $\| \|R(P)\|_1$ in the range $n > m^2$.

Lemma 2. *For a homogeneous polynomial P of degree m and $n > m^2 > 1$, we have*

$$(7) \quad \| \|R(P)\|_1 \leq \sqrt{2me \frac{n^{m-1}}{(m-1)!}} \|P\|_\infty.$$

Proof. We begin by observing that the number of monomials z^α in n variables of degree m with $\max_j \alpha_j > 1$ is $\binom{n+m+1}{m} - \binom{n}{m}$. Thus the Cauchy–Schwarz inequality gives

$$\| \|R(P)\|_1 \leq \sqrt{\left(\binom{n+m+1}{m} - \binom{n}{m} \right)} \|P\|_\infty.$$

The result follows from this because

$$\begin{aligned} \binom{n+m+1}{m} - \binom{n}{m} &\leq \frac{n^m}{m!} \left[\left(1 + \frac{m}{n}\right)^m - \left(1 - \frac{m}{n}\right)^m \right] \\ &\leq \frac{n^m}{m!} \left[e^{m^2/n} - e^{-m^2/n} \right] \leq 2em \frac{n^{m-1}}{(m-1)!}. \end{aligned}$$

\square

We turn next to the most challenging case, which is that of the tetrahedral part $T(P)$. Now the Cauchy–Schwarz inequality does not work because there are too many coefficients. We will transfer the problem to $T(B)$ and use instead a special form of the multilinear Khinchine inequality, which can be traced back to [5]. The precise formulation of the result to be used is

in [8, Theorem 3.2.2]. In the theorem below, $\{\epsilon_i\}_{i=1}^\infty$ denotes a Rademacher sequence of random variables, i.e., the ϵ_i are i.i.d and $\mathbb{P}(\epsilon_i = 1) = \mathbb{P}(\epsilon_i = -1) = 1/2$.

Theorem 3 (Hypercontractivity). *Let X be a homogeneous chaos of order m :*

$$X = \sum_{1 \leq i_1 < \dots < i_m \leq n} x_{i_1, \dots, i_m} \epsilon_{i_1} \cdots \epsilon_{i_m},$$

with $x_{i_1, \dots, i_m} \in \mathbb{C}$. Then

$$(\mathbb{E}(|X|^2))^{1/2} \leq e^m \mathbb{E}(|X|).$$

With this theorem we can prove the following.

Lemma 3. *For every homogeneous polynomial P of degree m with $m < n$, we have*

$$\|T(B)\|_1 \leq (e\kappa)^m \binom{n-1}{m-1}^{1/2} \|P\|_\infty,$$

where κ is the constant from Lemma 1.

Proof. Put

$$\begin{aligned} F &= \{(i_1, \dots, i_m), 1 \leq i_1, \dots, i_m \leq n \text{ pairwise distinct}\}, \\ F_{i_1} &= \{(i_2, \dots, i_m) : (i_1, i_2, \dots, i_m) \in F\}, \\ \tilde{F}_{i_1} &= \{(i_2, \dots, i_m) \in F_{i_1}, i_2 < \dots < i_m\}. \end{aligned}$$

We may write

$$\begin{aligned} \|T(B)\|_1 &= \sum_{(i_1, \dots, i_m) \in F} |B(e^{i_1}, \dots, e^{i_m})| \\ &= \sum_{i_1=1}^n \sum_{(i_2, \dots, i_m) \in F_{i_1}} |B(e^{i_1}, e^{i_2}, \dots, e^{i_m})| \\ &= \sum_{i_1=1}^n \sum_{(i_2, \dots, i_m) \in \tilde{F}_{i_1}} (m-1)! |B(e^{i_1}, e^{i_2}, \dots, e^{i_m})| \\ &\leq \binom{n-1}{m-1}^{1/2} \sum_{i_1=1}^n \left(\sum_{(i_2, \dots, i_m) \in \tilde{F}_{i_1}} ((m-1)! |B(e^{i_1}, e^{i_2}, \dots, e^{i_m})|)^2 \right)^{1/2}. \end{aligned}$$

By Theorem 3,

$$\begin{aligned} &\left(\sum_{(i_2, \dots, i_m) \in \tilde{F}_{i_1}} ((m-1)! |B(e^{i_1}, e^{i_2}, \dots, e^{i_m})|)^2 \right)^{1/2} \\ &\leq e^{m-1} \mathbb{E} \left(\left| \sum_{(i_2, \dots, i_m) \in \tilde{F}_{i_1}} (m-1)! B(e^{i_1}, e^{i_2}, \dots, e^{i_m}) \epsilon_{i_2} \cdots \epsilon_{i_m} \right| \right). \end{aligned}$$

Summing over i_1 , we get

$$(8) \quad \|T(B)\|_1 \leq e^{m-1} \binom{n-1}{m-1}^{1/2} \sup_{z \in \mathbb{D}^n} \sum_{i_1=1}^n \left| \sum_{(i_2, \dots, i_m) \in F_{i_1}} B(e^{i_1}, e^{i_2}, \dots, e^{i_m}) z_{i_2} \cdots z_{i_m} \right|.$$

We introduce the notation

$$\lambda_{i_1}(z) = \frac{\left| \sum_{(i_2, \dots, i_m) \in F_{i_1}} B(e^{i_1}, \dots, e^{i_m}) z_{i_2} \cdots z_{i_m} \right|}{\sum_{(i_2, \dots, i_m) \in F_{i_1}} B(e^{i_1}, \dots, e^{i_m}) z_{i_2} \cdots z_{i_m}}$$

and obtain from (8)

$$\begin{aligned} \|T(B)\|_1 &\leq e^{m-1} \binom{n-1}{m-1}^{1/2} \sup_{z \in \mathbb{D}^n} \sum_{(i_1, \dots, i_m) \in F} B(e^{i_1}, e^{i_2}, \dots, e^{i_m}) \lambda_{i_1}(z) z_{i_2} \cdots z_{i_m} \\ &\leq e^{m-1} \binom{n-1}{m-1}^{1/2} \sup_{(z^{(1)}, z^{(2)}) \in \mathbb{D}^n \times \mathbb{D}^n} |T(B)(z^{(1)}, \underbrace{z^{(2)}, \dots, z^{(2)}}_{m-1})| \\ &\leq e^m \binom{n-1}{m-1}^{1/2} \|T(P)\|_\infty \\ &\leq e^m \kappa^m \binom{n-1}{m-1}^{1/2} \|P\|_\infty, \end{aligned}$$

where in the last two steps we used Harris's inequality (6) and Lemma 1. \square

5. PROOF OF THEOREM 2

For the proof of Theorem 2, we need the following lemma of F. Wiener (see [2]).

Lemma 4. *Let P be a polynomial in n variables and $P = \sum_{m \geq 0} P_m$ its expansion in homogeneous polynomials. If $\|P\|_\infty \leq 1$, then $\|P_m\|_\infty \leq 1 - |P_0|^2$ for every $m > 0$.*

Proof of Theorem 2. We assume that $\sup_{\mathbb{D}^n} |\sum c_\alpha z^\alpha| \leq 1$. Observe that for all z in $r\mathbb{D}^n$,

$$\sum |c_\alpha z^\alpha| \leq |c_0| + \sum_{m > 1} r^m \sum_{|\alpha|=m} |c_\alpha|.$$

When $m > \log n$, we use (3) and Lemma 4, and obtain the estimate

$$(9) \quad \sum_{m > \log n} r^m \sum_{|\alpha|=m} |c_\alpha| \leq \sum_{m > \log n} r^m \sqrt{\binom{n+m-1}{m}} (1 - |c_0|^2),$$

whence

$$(10) \quad \sum_{m > \log n} r^m \sum_{|\alpha|=m} |c_\alpha| \leq \sum_{m > \log n} r^m (2e)^m \max(1, n/m)^{m/2} (1 - |c_0|^2).$$

If we take into account the estimate

$$\frac{(\log n)^m}{n} \leq m!$$

(obtained by a calculus argument), then Theorem 1 and Lemma 4 give

$$(11) \quad \sum_{m < \log n} r^m \sum_{|\alpha|=m} |c_\alpha| \leq \sum_{m < \log n} r^m \sqrt{m} C^m \left(\frac{n}{\log n} \right)^{m/2} (1 - |c_0|^2).$$

If we now choose $r \leq \varepsilon \sqrt{\frac{\log n}{n}}$ with ε small enough and combine (10) and (11), we obtain

$$\sum |c_\alpha z^\alpha| \leq |c_0| + (1 - |c_0|^2)/2 \leq 1$$

whenever $|c_0| \leq 1$. Thus the theorem is proved with $\gamma = \varepsilon$. \square

A closer examination of this proof shows that a better choice would be to use Theorem 1 only when $m < (2 + 2 \log \kappa)^{-1} \log n$. By this approach and taking into account the estimates from Lemmas 2 and 3, we get

$$b(n) \geq \frac{1}{\sqrt{2e(1 + \log \kappa)}} + o(1)$$

when $n \rightarrow \infty$. By also doing a meticulous analysis of (9) for “small” n and keeping in mind that $S(1, n) = 1$, one may arrive at a numerical value for γ which is larger than 0.2.

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